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On couple-stress fluid heated from below in porous medium

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Abstract : A layer of couple-stress fluid heated from below in porous medium is considered Using linearized stability theory and normal mode analysis, the dispersion relation is obtained For stationary convection, the couple-stress postpones the onset of convection whereas the medium permeability hastens the onset of convection The principle of exchange of stabilities is valid for the couple-stress fluid heated from below in porous medium.

Keywords : Couple-stress fluid, porous medium, principle of exchange of stabilities

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The study of a layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines, one can find the food process industry, chemical process industry, solidification and centrifugal casting of metals. The basic equations of a layer of fluid heated from below in porous medium, have been derived in a treatise by Joseph [1]. The use of the Boussinesq approximation has been made throughout which states that the variations of density in the equations of motion can be ignored everywhere except in its association with the external force. When the fluid permeates an isotropic and homogeneous porous medium, the gross effect is represented by the Darcy's law.

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Stokes [2] proposed and postulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrication of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid film is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, hip, knee and ankle joints are the loaded-bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear.

Normal synovial fluid is clear or yellowish and is a viscous, non-Newtonian fluid. According to the theory of Stokes [2], couple-stresses are found to appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [3] modeled synovial fluid as a couple-stress fluid in human joints.

Keeping in mind the importance of non-Newtonian fluids, convection in fluid layer heated from below and porous medium, the present paper is devoted to study the couple-stress fluid heated from below in porous medium.

Consider an infinite, horizontal, incompressible, couple-stress fluid layer of thickness d, heated from below, so that the temperatures and densities at the bottom surface $z = 0$ are T_0, ρ_0 and at the upper surface $z = d$ are T_d, ρ_d respectively, and that a uniform adverse temperature gradient $\beta\,(= |\,dT/dz\,|)$ is maintained. The gravity field $\mathbf{g}(0, 0, -g)$ pervades the system. This fluid layer is flowing through an isotropic and homogeneous porous medium of porosity ϵ and medium permeability k_1.

Let p, ρ, T and $\mathbf{q}(u, v, w)$ denote respectively the fluid pressure, density, temperature and filter velocity. Then the momentum balance, mass balance and energy balance

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equations of couple-stress fluid through porous medium [1,2] are

$$\frac{1}{\epsilon}\left[\frac{\partial \boldsymbol{q}}{\partial t}+\frac{1}{\epsilon}(\boldsymbol{q}.\nabla)\boldsymbol{q}\right]=-\frac{1}{\rho_0}\nabla p+\boldsymbol{g}\left(1+\frac{\delta\rho}{\rho_0}\right)$$

$$-\frac{1}{k_1}\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\boldsymbol{q}, \tag{1}$$

$$\nabla.\boldsymbol{q}=0, \tag{2}$$

$$E\frac{\partial T}{\partial t}+(\boldsymbol{q}.\nabla)T=\kappa\nabla^2 T. \tag{3}$$

The equation of state is

$$\rho=\rho_0\left[1-\alpha(T-T_0)\right], \tag{4}$$

where the suffix zero refers to values at the reference level $z = 0$. The kinematic viscosity ν, couple-stress viscosity μ', thermal diffusivity κ and coefficient of thermal expansion α are all assumed to be constants. $E=\epsilon+(1-\epsilon)\frac{\rho_s c_s}{\rho_0 c_f}$ is constant where ρ_s, c_s and ρ_0, c_f stand for density and specific heat of solid (porous matrix) material and fluid respectively.

The basic motionless solution is

$$\boldsymbol{q} = (0, 0, 0),\ T = T_0 - \beta z,\ \rho = \rho_0(1 + \alpha\beta z). \tag{5}$$

Here, we use linearized stability theory and normal mode analysis method. Assume small perturbations around the basic solution and let δp, $\delta\rho$, θ and $\boldsymbol{q}(u, v, \omega)$ denote respectively the perturbations in pressure p, density ρ, temperature T and velocity (0, 0, 0). The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature, is given by

$$\delta\rho=-\alpha\rho_0\theta. \tag{6}$$

Then the linearized perturbation equations of the couple-stress fluid become

$$\frac{1}{\epsilon}\frac{\partial \boldsymbol{q}}{\partial t} \quad \frac{1}{\rho_0}\nabla\delta p-g\alpha\theta-\frac{1}{k_1}\ \ \nu-\frac{\mu}{\rho_0}\nabla^2\ \ |\boldsymbol{q}, \tag{7}$$

$$\nabla.\boldsymbol{q}=0, \tag{8}$$

$$E\frac{\partial\theta}{\partial t}=\beta\omega+\kappa\nabla^2\theta. \tag{9}$$

Writing the scalar components of eq. (7) and eliminating u, v, δp between them by using (8), we obtain

$$\frac{1}{\epsilon}\frac{\partial}{\partial t}\nabla^2\omega+\frac{1}{k_1}\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\omega$$

$$-g\alpha\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)=0 \tag{10}$$

Eq. (9) can be re-written as

$$\left(E\frac{\partial}{\partial t}-\kappa\nabla^2\right)\theta=\beta\omega. \tag{11}$$

Analysing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$\left[\omega,\theta\right]=\left[W(z),\Theta(z)\right]\exp(ik_x x+ik_y y+nt), \tag{12}$$

where k_x, k_y are wave numbers along the x- and y- directions respectively, $k\left(=\sqrt{k_x^2+k_y^2}\right)$ is the resultant wave number and n is, in general, a complex constant.

Expressing the coordinates x, y, z in the new unit of length d and letting $a = kd$, $\sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$, $P_l=\frac{k_1}{d^2}$, $F=\frac{\mu'/\rho_0 d^2}{\nu}$ and $D=\frac{d}{dz}$; Eqs. (10) and (11), using (12), yield

$$\left(D^2-a^2\right)\left[1+\frac{\sigma}{\epsilon}P_l-F\left(D^2-a^2\right)\right]W$$

$$=-\frac{g\alpha d^2}{\nu}a^2 P_l\Theta, \tag{13}$$

$$\left(D^2-a^2-Ep_1\sigma\right)\Theta=-\ \beta d^2\ W \tag{14}$$

Since both boundaries are maintained at constant temperatures, the perturbations in temperature are zero at the boundaries. Consider the case of two free boundaries. The case of two free boundaries is a little artificial but it enables us to find analytical solutions and to make some qualitative conclusions. The appropriate boundary conditions, with respect to which eqs. (13) and (14) must be solved, are

$$W = D^2W = 0,\ \Theta = 0 \qquad \text{at } z = 0 \text{ and } 1. \tag{15}$$

Using the boundary conditions (15), it can be shown that all the even order derivatives of W must vanish for $z = 0$ and 1 and hence the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{16}$$

where W_0 is a constant.

Eliminating Θ between eqs. (13) and (14) and substituting the proper solution (16) in the resultant equation, we obtain the dispersion relation

$$R_1 = \frac{1+x}{xP}(1+x+iEp_1\sigma_1)\left(1+\frac{iP}{\epsilon}\sigma_1+F\pi^2\overline{1+x}\right), \tag{17}$$

where $R_1=\frac{R}{\pi^4}=\frac{g\alpha\beta d^4}{\nu\kappa\pi^4}$, $x=\frac{a}{\pi^2}$, $i\sigma_1=\frac{\sigma}{\pi^2}$ and $P=\pi^2 P_l$

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (17) reduces to

$$R_1=\frac{(1+x)^2}{xP}(1+F\pi^2\overline{1+x}). \tag{18}$$

To study the effects of couple-stress parameter and medium permeability, we examine the natures of $\frac{dR_1}{dF}$ and $\frac{dR_1}{dP}$ analytically. Eq. (18) yields

$$\frac{dR_1}{dF}=\frac{\pi^2(1+x)^3}{xP} \tag{19}$$

$$\frac{dR_1}{dP} = -\frac{(1+x)^2}{xP^2}\cdot\left(1+F\pi^2\overline{1+x}\right). \tag{20}$$

Thus for stationary convection, the couple-stress postpones the onset of convection whereas the medium permeability hastens the onset of convection, on the couple-stress fluid heated from below in porous medium.

Multiplying eq. (13) by W^*, the complex conjugate of W, and using (14) together with the boundary conditions (15), we obtain

$$FI_1 + \left(1 + P_l\frac{\sigma}{\epsilon}\right)I_2 = \frac{g\alpha\kappa a^2}{\nu\beta}P_l\left(I_3 + Ep_1\sigma^* I_4\right), \tag{21}$$

where $I_1 = \int_0^1\left(|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2\right)dz$,

$$I_2 = \int_0^1\left(|DW|^2 + a^2|W|^2\right)dz,$$

$$I_3 = \int_0^1\left(|D\Theta|^2 + a^2|\Theta|^2\right)dz, \tag{22}$$

$$I_4 = \int_0^1 |\Theta|^2 dz,$$

and are all positive definite.

The real and imaginary parts of eq. (21) must vanish separately; and the vanising of the imaginary part gives

$$im(\sigma)\left|\frac{I_2}{\epsilon} + \frac{g\alpha\kappa a^2}{\nu\beta}Ep_1 I_4\right| = 0 \tag{23}$$

But the quantity inside the brackets is positive definite. Hence,

$$im(\sigma) = 0. \tag{24}$$

This establishes that σ is real and that the principle of exchange of stabilities is valid for the couple-stress fluid heated from below in porous medium.

References

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